

Assume  $P(X, Y, Z) = X^2 + Y^2 + Z^2$  to be reducible in  $k[X, Y, Z]$ . We can then write:  $P(X, Y, Z) = (X + Q(Y, Z))(X + R(Y, Z))$ , which implies that  $R(Y, Z) + Q(Y, Z) = 0$ , and  $Q(Y, Z)R(Y, Z) = Y^2 + Z^2$ . Then  $-R(Y, Z)^2 = Y^2 + Z^2$ , which is clearly impossible if  $\text{char}(k) \neq 2$ . Why? Because we would have  $R(Y, Z) = YR_1(Z) + R_2(Z)$ , and  $-R(Y, Z)^2 = -Y^2R_1(Z)^2 - R_2(Z)^2 - 2YR_1(Z)R_2(Z) = Y^2 + Z^2$ , which amounts to having  $R_1(Z)^2 = -1$ ,  $2R_1(Z)R_2(Z) = 0$ , and  $R_2(Z)^2 = Z^2$ . This is of course impossible if  $\text{char}(k) \neq 2$ , whence  $P(X, Y, Z)$  is irreducible in  $k[X, Y, Z]$  in this case. If  $\text{char}(k) = 2$ ,  $R_1(Z) = 1$  and  $R_2(Z) = Z$  is the only solution. In this case,  $P(X, Y, Z) = (X + Y + Z)^2$ , whence  $P$  is not irreducible.

Let  $P(X, Y, Z) = X^n + Y^n + Z^2$ . By the same method used above,  $P$  is irreducible if  $n$  is odd.

If  $n = 0$  then  $P$  is reducible over  $k$  if and only if  $k$  contains a square root of  $-2$  (for instance  $k = \mathbf{C}$ , or any field  $k$  with characteristic 2).

If  $n \geq 2$  is even ( $n = 2n'$ , with  $n' \geq 1$ ), then we must find  $R(Y, Z)$  under the form  $Y^{n'} + Z^{n'} + T(Y, Z)$ , i.e  $T(Y, Z)$  must satisfy  $T^2(Y, Z) + 2Y^{n'}Z^{n'} + 2(Y^{n'} + Z^{n'})T(Y, Z) = 0$  (\*). Whence  $T(Y, Z)$  must divide  $2Y^{n'}Z^{n'}$ , i.e  $T(Y, Z) = aY^uZ^v$  for some  $u, v \leq n'$ . Substituting in (\*) and solving shows that there is a solution if and only if  $\text{char}(k) = 2$ , and  $T(X, Y) = 0$ . This means  $P$  is irreducible if and only if  $\text{char}(k) = 2$ . For instance, in  $\mathbf{Z}/2\mathbf{Z}[X]$ ,  $X^n + Y^n + Z^2 = (X^{n'} + Y^{n'} + Z)^2$ .

In conclusion, for  $n \geq 1$ ,  $P$  is irreducible if and only if  $\text{char}(k) \neq 2$ . For  $n = 0$ ,  $P$  is irreducible if and only if  $-2$  has no square root in  $k$ .