By the mean value theorem, $\exists x \in]0,1[, f'(x) = 1$. Whence if $f'(0) = 1$, we are done. Otherwise, we can assume $f'(0) > 1$ without loss of generality. It is then fairly clear that $f'(z) < 1$ for some $z \in [0,1]$ (otherwise we would have $f(1) > 1$). But f' satisfies the intermediate value property by Darboux's theorem, which makes it easy to conclude (just pick $\epsilon > 0$ small enough so that $1+\epsilon, \frac{1}{1+\epsilon}$ are both in $[f'(z), f'(0)]$, and apply Darboux's theorem).