Since $f+g$ is nondecreasing, it admits a left and a right limit at each point (except at 0 and at 1, where $f+g$ has a right (resp. left) limit; if $x=0$, then we set $f(x-)=f(0)$; likewise, we define $f(x+)=f(1)$), and f has the same property by continuity of g.

If f is discontinuous at point x, then necessarily $f(x+)>f(x-)$ (*). The reason for this is that otherwise we would have $f(x+)\leq f(x-)$ and $f(x+)+g(x+)=f(x+)+g(x)\leq f(x-)+g(x)=f(x-)+g(x-)$. This amounts to saying that $(f+g)(x+)\leq(f+g)(x-)$, which is absurd (the equality case is easily discarded by monotonicity of $f+g$). Note also that we must have $(f+g)(x+)\geq(f+g)(x)\geq(f+g)(x-)$, which implies that $f(x+)\geq f(x)\geq f(x-)$ (**).

Let $z=inf(E=\{x;\ f(x)\leq 0\})$. If $f(z)=0$, the problem is solved.
If $f(z)<0$, then necessarily $f(z-)\geq 0$ (by definition of the inf), which is absurd by (**).
Thus, $f(z)>0$. But then $f(x+)<0$ (because z is an accumulation point of E), while $f(x-)\geq 0$. This is absurd by (*).