Since ff' has no root on $[0,1]$, f is either positive or negative on the whole interval. Without loss of generality, we can assume $f > 0$ (otherwise consider $-f$). Similarly, f is either increasing or decreasing on $[0,1]$; without loss of generality, we can assume f is increasing (otherwise consider $f(1-x)$). Now we have either $f'' \geq 1$ or $f'' \leq -1$ on $[0,1]$ (this is because of Darboux's theorem).

Assume $f'' \geq 1$, then $f'(x) \geq x + f'(0)$, and $f(x) \geq f(0) + \frac{x^2}{2} + f'(0)x$. Taking $x = 1$ yields the result.

If we assume $f'' \leq -1$, then $f'(1) \leq -1 + u + f'(u)$, and since $f'(1) > 0$, we must have $f'(u) > 1 - u$, and thus $f(u) > u - \frac{u^2}{2} + f(0)$, which gives $f(1) > \frac{1}{2}$.