Actually, $[0,1]$ can be replaced by any compact metric space (X,d). Take $f_n(x) = \sup_y (f(y) - nd(x,y))$. The continuity of f_n is obvious, $f_n(x) \geq f(x)$ and $f_n(x) - f(x) = \sup_y ([f(y) - f(x)] - nd(x,y)) \geq 0$. Since f has a finite sup on M (we will prove this latter), as n becomes larger, the sup will be attained for some $|y - x| \leq \epsilon_n$, where $\epsilon_n \to 0$. This guarantees, by upper semicontinuity of f, that $|f_n(x) - f(x)|$ goes to 0 as $n \to +\infty$.

Let's prove that f has a finite sup on X. Assuming the contrary, you can construct a sequence x_n which converges to a point $x \in M$ and such that $f(x_{n+1}) - f(x_n) > 1$. This contradicts the upper semi-continuity of f at point x. Note that we can actually say more: the sup of f on X must be attained at x.